# AN EXAMPLE OF THE INDICATRIX IN THE CALCULUS OF VARIATIONS.

By ARNOLD DRESDEN, The University of Chicago.

## § 1. INTRODUCTION.

Suppose there is given a definite integral

$$I=\int_{t_1}^{t_2} F(x,\ y,\ x',\ y')\,dt, \tag{1}$$

in which $x$ and $y$ are functions of some parameter $t$, $x'$ and $y'$ are the derivatives of these functions with respect to $t$. Let the function $F$ be continuous and have continuous derivatives of the first, second and third order in a domain $T$ of the variables $x$, $y$, $x'$, $y'$, defined by $(x, y)$ in a region $R$ of the $xy$-plane, $x'$ and $y'$ finite and restricted by the condition

$$x'^2+y'^2\neq 0. \tag{2}$$

The definite integral (1) may now be taken along an infinitude of curves between $P_1[x(t_1),\ y(t_1)]$ and $P_2[x(t_2),\ y(t_2)]$ in the domain $T$.

The simplest problem of the Calculus of Variations is to determine among the totality of all these curves, restricted by certain conditions, the one for which $I$ is a *maximum*, or a *minimum**. We shall use the word *extremum* to denote either maximum or minimum.

Let now

$$x=\phi(t),\quad y=\psi(t),\quad t_1 \overline{\gtrless} t \overline{\gtrless} t_2, \tag{3}$$

be the equations in parameter-representation† of a curve which minimizes (1). If we restrict ourselves to functions $\phi$ and $\psi$, which have continuous first derivatives, the conditions that (3) furnishes a weak‡ minimum for (1) are§

1. *The functions $\phi(t)$ and $\psi(t)$ must satisfy Euler's differential equation, in the Weierstrass-form, i. e.,*

$$\overline{F}_{x'y}-\overline{F}_{xy'}+\overline{F}_1[\phi''\psi'-\phi'\psi'']=0. \tag{I}$$

The function $F_1$ is defined by

$$F_1(x,\ y,\ x',\ y')=\frac{F_{x'x'}(x,\ y,\ x',\ y')}{y'^2}=-\frac{F_{x'y'}(x,\ y,\ x',\ y')}{x'y'}=\frac{F_{y'y'}(x,\ y,\ x',\ y')}{x'^2}. \tag{4}$$

*For an exact formulation of the problem see O. Bolza, Lectures on the Calculus of Variations, §3.

†See C. Jordan, Cours d'Analyse, Vol. I, 2nd ed., pp. 90-108.

‡See O. Bolza, loc. cit., pp. 69-71.

§Ibid., Chapter IV.

The stroke over the function-symbols is used to denote that the arguments are to be taken as follows:

$$x=\phi(t),\quad y=\psi(t),\quad x'=\phi'(t),\quad y'=\psi'(t).$$

Any curve, satisfying (I), will be called an *extremal.**

II. *Legendre's condition must be fulfilled, i. e.,*

$$\bar{F}_1 \geqq 0,\quad t_1 \overline{\gtrless} t \overline{\gtrless} t_2.$$

III. *Jacobi's condition must be satisfied, i. e.,*

$$t_2 \leqq t_1'.$$

$t_1'$ denotes here the parameter-value of the conjugate-point† of $P(t_1)$.

If the minimum shall be strong‡, a fourth condition needs to be satisfied:

IV. $E(x, y; x' y'; \bar{x}', \bar{y}') \geqq 0$, for every point along (3) and for any pair of finite values of $\bar{x}'$, $\bar{y}'$ different from $x'$, $y'$, and restricted by the condition:

$$\bar{x}'^2+\bar{y}'^2\neq 0.$$

The function $E$ is defined by:

$$E(x, y; x', y'; \bar{x}', \bar{y}')=\bar{x}'[F_{x'}(x, y, \bar{x}', \bar{y}')-F_{x'}(x, y, x', y')]+$$
$$+\bar{y}'[F_{y'}(x, y, \bar{x}', \bar{y}')-F_{y'}(x, y, x', y')].$$

A stronger form of (IV) is

(IV') $\qquad F_1(x, y, \cos\gamma, \sin\gamma) \geqq 0$ along (3), and for $0\leqq\gamma\leqq 2\pi$.

A curve, satisfying condition (IV'), shall be called a *hyperstrong minimum.*

The conditions for a weak, strong, or hyperstrong maximum are obtained out of (I), (II), (III), (IV), and (IV') by replacing the inequality signs by the opposite ones.

The conditions, as stated above, with the inclusion of equality signs, are the conditions for *improper extrema.*§

By omitting the equality signs from (II), (III), (IV), and (IV'), we obtain the conditions for *proper extrema.*§

If we free ourselves of the restriction, that the functions $\phi'(t)$ and $\psi'(t)$ should be continuous, and admit curves with corners (so called *discon-*

*See O. Bolza, loc. cit., p. 27, p. 123.
†See O. Bolza, loc. cit., p. 60, p. 135.
‡See O. Bolza, loc. cit., p. 69, p. 71.
§See O. Bolza, loc. cit., p. 11.

*tinuous solutions**) as solutions of our problems, still another condition must be satisfied:
(V) *Weierstrass' corner-condition.*† At every corner we must have:

$$\left.\begin{aligned} F_{x'}(x, y, x', y') &= F_{x'}(x, y, \underline{x}', \underline{y}') \\ F_{y'}(x, y, x', y') &= F_{y'}(x, y, \underline{x}', \underline{y}') \end{aligned}\right\}, \tag{V}$$

where $x'$, $y'$, and $\underline{x}'$, $\underline{y}'$ denote the forward and backward derivatives of $\phi(t)$ and $\psi(t)$ at the corner.

2. Carathéodory‡ has given a method by means of which we can decide whether or not the conditions (II), (IV), (IV'), and (V) are satisfied by a given extremal.

For every point of the region $R$, in which the function $F$, and its derivatives of first, second, and third order are determined, he defines a curve, called the *Indicatrix*, by means of its equation in polar coordinates:

$$\rho = \frac{1}{F(x, y, \cos\theta, \sin\theta)}, \tag{5}$$

the origin of coordinates being a point $G$. We make the usual convention of Analytic Geometry, that, if $\rho < 0$, for $\theta = \theta_1$, the absolute value of $\rho$ shall be laid off on the half-line $\theta = \theta_1 + \pi$.

He proves then the following theorems:§

*A*. If $F(x, y, \cos\theta, \sin\theta)$ is of constant sign for $0 \leqq \theta \leqq 2\pi$, the indicatrix is a closed curve around the origin $G$.

*Proof for A*. The theorem follows at once from the definition of the Indicatrix and the usual conventions concerning polar coordinates.

*B*. If $F(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0$, and the indicatrix has positive curvature at $\theta = \theta_0$,

$$F_1(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0.$$

If $F(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0$, and the indicatrix has negative curvature at $\theta = \theta_0$,

$$F_1(x, y, \cos\theta_0, \sin\theta_0) \lesseqgtr 0.$$

*Proof for B*. Referring the Indicatrix to rectangular coordinates:

$$\xi = \rho\cos\theta, \quad \eta = \rho\sin\theta,$$

we find

*See O. Bolza, loc. cit., p. 36.

†Ibid., p. 38, p. 126.

‡C. Caratheodory, *Ueber die discontinuierlichen Loesungen in der Variationsrechnung*, Gottingen, 1904, p. 69; *Mathematische Annalen*, Vol. 62, p. 456.

§*Mathematische Annalen*, Vol. 62, pp. 457, 460, 461, 465.

$$\xi=\frac{\cos\theta}{F(x,\ y,\ \cos\theta,\ \sin\theta)},\quad \eta=\frac{\sin\theta}{F(x,\ y,\ \cos\theta,\ \sin\theta)}. \tag{6}$$

From this, making use of the formulae:

$$F(x,\ y,\ \cos\theta,\ \sin\theta)=F_{x'}\cos\theta+F_{y'}\sin\theta,$$

$$F'(x,\ y,\ \cos\theta,\ \sin\theta)=-F_{x'}\sin\theta+F_{y'}\cos\theta,^{*}$$

and the definition of the $F_1$-function, we obtain:

$$\xi'=-\frac{F_{y'}}{F^2},\quad \eta'=\frac{F_{x'}}{F^2} \tag{7}$$

and

$$\xi''=\frac{2F'F_{y'}-FF_1\cos\theta}{F^3},\quad \eta''=-\frac{2F'F_{x'}+FF_1\sin\theta}{F^3},$$

where $'$ indicates differentiation with respect to $\theta$. Consequently:

$$\frac{d^2\xi}{d\eta^2}=\frac{\xi'\eta''-\xi''\eta'}{\xi'^2}=\frac{F_1}{\xi'^2F^3};$$

from which the theorem follows at once.

*C.* If $F(x,\ y,\ \cos\theta_0,\ \sin\theta_0)\gtrless 0$, and $G$ and $\bar{Q}$ lie on the same side of the tangent to the indicatrix at $Q(\theta_0)$,

$$E(x,\ y;\ \cos\theta_0,\ \sin\theta_0;\ \cos\bar{\theta}_0,\ \sin\bar{\theta}_0)\gtrless 0.$$

If $F(x,\ y,\ \cos\theta_0,\ \sin\theta_{\bar{0}})\gtrless 0$, and $G$ and $\bar{Q}(\bar{\theta}_0)$ lie on opposite sides of the tangent to the indicatrix at $Q(\theta_0)$,

$$E(x,\ y;\ \cos\theta_0,\ \sin\theta_0;\ \cos\bar{\theta}_0,\ \sin\bar{\theta}_0)\lesseqgtr 0$$

(see Fig. 1).

*Proof for C.* (See Fig. 1.) From equations (6) and (7) follows

$$F_{x'}X+F_{y'}Y=1, \tag{8}$$

as the equation for the tangent at a point $\theta=\theta_0$ to the Indicatrix for the point $(x_0,\ y_0)$, when $X$ and $Y$ are running coordinates, and the arguments of $F_{x'}$ and $F_{y'}$ are $x_0,\ y_0,\ \cos\theta_0,\ \sin\theta_0$.

For the perpendicular from a point $\bar{Q}(\bar{\theta})$ of the Indicatrix on the tangent at a point $Q(\theta)$, we find:

*See Bolza, loc. cit., p. 120.

$$\bar{Q}M = \frac{-E\,(x_0, y_0;\ \cos\theta, \sin\theta;\ \cos\bar{\delta}, \sin\bar{\delta})}{F(x_0, y_0, \cos\bar{\theta}, \sin\theta)\ \sqrt{(F^2_{x'} + F^2_{y'})}},$$

while for the perpendicular from the origin $G$ on the same tangent, we obtain

$$GM_G = \frac{-1}{\sqrt{(F^2_{x'} + F^2_{y'})}} < 0.$$

From the usual convention concerning the sign of a perpendicular, we obtain then:

$\bar{Q}M < 0$, if $\bar{Q}$ and $G$ are on the same side of $QM$,

$\bar{Q}M > 0$, if $\bar{Q}$ and $G$ are on opposite sides of $QM$,

from which the theorem is at once evident.

*D.* If the indicatrix for a point $(x_0, y_0)$ admits a double-tangent, touching the curve at the points $A(\alpha)$ and $A'(\alpha')$, the point $(x_0, y_0)$ is the corner of a possible discontinuous solution, the direction of the branches being $\alpha$ and $\alpha'$.

*Proof for D.* From the equation for the tangent to the Indicatrix, (8), and from No. (V) of the general theorems (p. 121), we conclude that if the corner-condition is satisfied at a point $(x_0, y_0)$ for two directions $\alpha$ and $\alpha'$, the tangents at the points $A(\alpha)$ and $A(\alpha)$ to the Indicatrix for the point $(x_0, y_0)$ must be coincident, From this, the theorem follows immediately.

After these remarks, we can now go over to the subject proper of this paper, the treatment of a simple problem of the Calculus of Variations, making use of the Indicatrix. The problem treated was given by Caratheadory himself.*

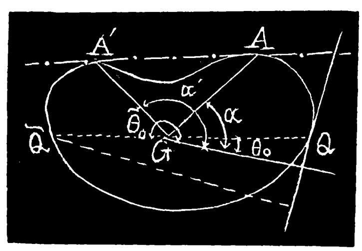


Fig. 1.

3. It is required to minimize the definite integral:

$$I = \int_{t_1}^{t_2} \left[ \frac{\sqrt{[x'^2(y^2+1) - 2xyx'y' + y'^2(x^2+1)]}}{x^2+y^2+1} - \frac{\sqrt{(x'^2+y'^2)}}{4} \right] dt.$$

We have then:

$$F(x, y, x', y') = \frac{\sqrt{[x'^2(y^2+1) - 2xyx'y' + y'^2(x^2+1)]}}{x^2+y^2+1} - \frac{\sqrt{(x'^2+y'^2)}}{4},$$

and find

$$F_{x'y} = F_{xy'}, \quad F_1 = \frac{1}{[1+(xy'-x'y)^2]^{\frac{3}{2}}} - \tfrac{1}{4}.$$

---

*C. Caratheodory, Ueber die disc. Loesungen, etc., p. 38.

(a) Euler's differential equation becomes

$$(x'y''-x''y')F=0.$$

The general integral of this is $x=my+n$, which represents the straight lines of the plane. We obtain also a singular solution from $F_1=0$,

$$1+(xy'-x'y)^2=4^{2/3}, \quad xy'+x'y=(4^{2/3}-1)^{\frac{1}{2}}.$$

We choose now the arc length $s$, as our functional parameter, and make the following transformation of coordinates (see Fig. 2):

$$\left.\begin{array}{ll} x=r\cos\phi, & x'=\cos\theta \\ y=r\sin\phi, & y'=\sin\theta \\ \multicolumn{2}{c}{\psi=\theta-\phi} \end{array}\right\}. \tag{9}$$

The singular integral becomes then

$$r\sin\psi=(4^{2/3}-1)^{\frac{1}{2}},$$

also representing a straight line.

The solutions of Euler's differential equation called *extremals* prove to be the straight lines of the plane.

In the sequel, we denote by $a$, the constant $\sqrt{(4^{2/3}-1)}$.

Applying the transformations (9) to the functions $F(x, y, x', y')$, and $F_1(x, y, x', y')$, we get

$$\left.\begin{array}{l} \mathrm{F}=\dfrac{\sqrt{[1+r^2\sin^2\psi]}}{r^2+1}-\frac{1}{4}, \\[2ex] \mathrm{F}_1=\dfrac{1}{[1+r^2\sin^2\psi]^{\frac{3}{2}}}-\frac{1}{4}. \end{array}\right. \tag{10}$$

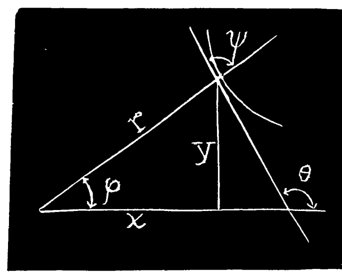


Fig. 2.

(b) For Legendre's condition, we have to consider the sign of $\mathrm{F}_1$. We find from (10 a):

$$\mathrm{F}_1>0, \text{ if } r\sin\psi<a, \quad \mathrm{F}_1<0, \text{ if } r\sin\psi>a.$$

It follows, that the straight lines, which intersect the circle of radius $a$ (denoted by $C_a$, see Fig. 3,, are minima, and those lying outside $C_a$ are maxima.

(c) The extremals being straight lines, it follows from the geometrical interpretation of the conjugate point,* that Jacobi's condition is fulfilled by every straight line of the plane.

*See Bolza, loc. cit., pp. 60-63, p. 137.

We have then the following result:

I. *Every straight line in the plane intersecting $C_a$ furnishes at least a weak minimum.*

*Every straight line in the plane, lying outside $C_a$, furnishes at least a weak maximum.*

The straight lines which are tangent to $C_a$ form a limiting case which will be considered later.

(d) We find:

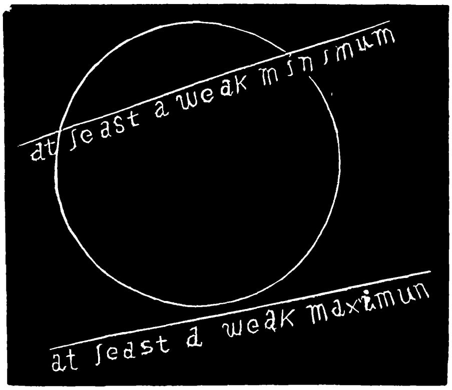

Fig. 3.

$$E(x,\ y;\ \cos\theta,\ \sin\theta;\ \cos\bar\theta,\ \sin\bar\theta)=$$

$$=\frac{1+r^2\sin^2\bar\psi}{\sqrt{[1+r^2\sin^2\bar\psi]\,(r^2+1)}}-\frac{\cos(\psi-\bar\psi)+r^2\sin.\psi\ \sin.\bar\psi}{\sqrt{[1+r^2\sin^2\psi]\,(r^2+1)}}+\frac{\cos(\psi-\bar\psi)-1}{4},$$

where (7) is applied and $\bar\psi=\bar\theta-\phi$.

For the further investigation, we have to discuss the sign of this function, which is a very cumbersome problem. At this point, we introduce the Indicatrix for this problem, by means of which the remaining questions can be more readily answered.

4. The *Indicatrix* is determined by the equation:

$$\rho=\frac{1}{\sqrt{\dfrac{(1+r^2\sin^2\psi)}{r^2+1}}-\frac{1}{4}}=\frac{1}{F(r,\ \psi)}. \tag{11}$$

$r\equiv\sqrt{(x^2+y^2)}$ functions here as a parameter, which takes all real positive values, thus furnishing a curve for every point of the plane.

For the character of the Indicatrix it is of importance to know the sign of $F(r,\ \psi)$ and $F_1(r,\ \psi)$ for all values of $\psi$ between 0 and $2\pi$.* Since $\frac{\partial F}{\partial\psi}>0$, for $0\leqq\psi\leqq\frac{1}{2}\pi$, we have:

$$F_{\text{min.}}=F(r,\ 0)=\frac{1}{1+r^2}-\frac{1}{4},$$

$$F_{\text{max.}}=F(r,\ \tfrac{1}{2}\pi)=\frac{1}{\sqrt{(1+r^2)}}-\frac{1}{4}.$$

We conclude:

*See page 121, Theorems *A* and *B*.

F *constantly positive,* if $F_{min.}>0$, *i. e.*, if $\frac{1}{1+r^2}>\frac{1}{4}$ or $r<\sqrt{3}$.

F *constantly negative,* if $F_{max.}<0$, *i. e.*, if $\frac{1}{\sqrt{(1+r^2)}}<\frac{1}{4}$ or $r<\sqrt{15}$.

F *varying,* if $F_{min.}<0$, $F_{max.}>0$, *i. e.*, if $\sqrt{3}<r<\sqrt{15}$.*

We have previously found, that:

$$F_1>0, \text{ if } r\sin\psi<a.$$
$$F_1<0, \text{ if } r\sin\psi>a.$$

These results show, that the character of the Indicatrix will be essentially different for points lying in one of the four regions, into which the plane is divided by the three circles of radius $a$, $\sqrt{3}$, and $\sqrt{15}$, respectively (see Fig. 4, circles $C_a$, $C_3$, and $C_{15}$).

The problem is now reduced to the discussion of the properties of the Indicatrix in each of the four cases:

I. $0<r<a$.
II. $a<r<\sqrt{3}$.
III. $\sqrt{3}<r<\sqrt{15}$.
IV. $\sqrt{15}<r$,

after which the limiting cases $r=a$, $\sqrt{3}$, and $\sqrt{15}$, respectively, still have to be considered.

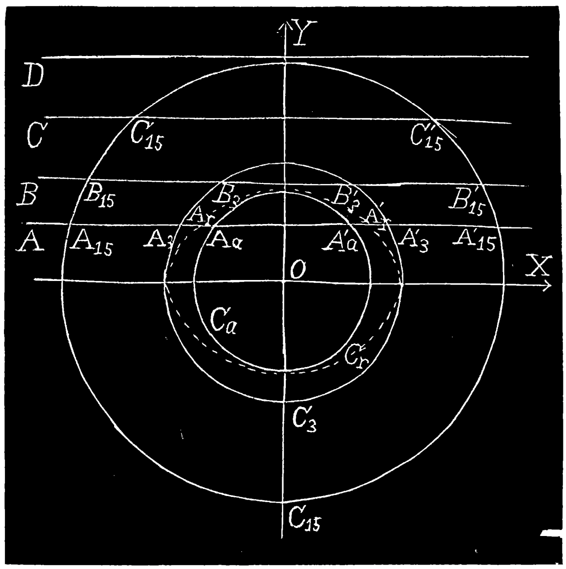


Fig. 4.

( To be continued. )

*$r=\sqrt{3}$ and $r=\sqrt{15}$ are two limiting cases, which will be considered later.